Topic 10 - Reduction of order

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Suppose you know one solution y_1 to the homogeneous ODE

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0} \qquad (*)$$

on an interval I where $y_1(x) \neq 0$ on I. Then one can find another solution using

$$\boxed{y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx}$$

Further, y_1 and y_2 will be linearly independent. Thus,

$$y_h = c_1 y_1 + c_2 y_2$$

will give all solutions to $(*)$.

Proof: The derivation of this formula is at the end of these notes. $\blacksquare$

Ex: Consider

$$(x^2+1)y'' - 2xy' + 2y = 0 \qquad (**)$$

on $I = (0, \infty)$.

Let $y_1 = x$.

Then $y_1 = x$ solves $(**)$ since if you plug it in you get

$$(x^2+1)\cdot 0 - 2x(1) + 2(x) = 0$$

Let's find our second solution.

First we must put a 1 in front of y'' in $(**)$.

Divide by (x^2+1) to get

$$y'' \underbrace{- \frac{2x}{x^2+1}}_{a_1(x)} y' + \frac{2}{x^2+1} y = 0$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x \int \frac{e^{-\int -\frac{2x}{x^2+1}dx}}{(x)^2}\,dx$$

$$= x \int \frac{e^{\int \frac{2x}{x^2+1}dx}}{x^2}\,dx$$

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du = \ln|u|$$

$$= \ln|x^2+1|$$

$$= \ln(x^2+1)$$

$u = x^2+1$
$du = 2x\,dx$

$x^2+1>0$

$$= x \int \frac{e^{\ln(x^2+1)}}{x^2}\, dx$$

$$= x \int \frac{x^2+1}{x^2}\, dx$$

$$e^{\ln|A|} = A$$

$$= x \int \left(\frac{x^2}{x^2} + \frac{1}{x^2} \right) dx$$

$$= x \int \left(1 + x^{-2} \right) dx$$

$$= x \left(x + \frac{x^{-1}}{-1} \right)$$

$$= x \left(x - \frac{1}{x} \right)$$

$$= x^2 - 1$$

So, $y_1 = x$, $y_2 = x^2 - 1$.

Thus, the general solution to

$$(x^2+1)y'' - 2xy' + 2y = 0$$

is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x + c_2 (x^2 - 1) \end{aligned}$$

Ex:

Given that $y_1 = x^4$ is a solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$, find the general solution.

First divide by x^2 to put a 1 in front of y''.

We get

$$y'' \underbrace{- \frac{7}{x}}_{a_1(x) = -\frac{7}{x}} y' + \frac{16}{x^2} y = 0$$

We get

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x}dx}}{(x^4)^2}\,dx$$

$$= x^4 \int \frac{e^{\int \frac{7}{x}dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x}dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$$\boxed{\int \frac{1}{x}dx = \ln|x| = \ln(x) \quad x>0}$$

$I = (0, \infty)$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$A\ln(B)$
$= \ln(B^A)$

$$= x^4 \int \frac{x^7}{x^8}$$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln|x|$$

$I = (0, \infty)$
$x > 0$

$$= x^4 \ln(x)$$

Thus the general solution to

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$ is

$$y_h = c_1 y_1 + c_2 y_2$$
$$= c_1 x^4 + c_2 x^4 \ln(x)$$

We now give a proof/derivation of the formula that we have been using

Suppose that y_1 is a known solution to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0 \quad (*)}$$

and assume $y_1(x) \neq 0$ for all x in I.

Let $y_2(x) = v(x) \cdot y_1(x)$.

We want to find v so y_2 also solves $(*)$.

We know by assumption that

$$y_1'' + a_1(x)y_1' + a_0(x)y_1 = 0$$

Since $y_2 = v \cdot y_1$ we get

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$$

$$= v''y_1 + 2v'y_1' + vy_1''$$

Subbing these into (*) we want to find v such that

$$(v''y_1 + 2v'y_1' + vy_1'') + a_1(x)(v'y_1 + vy_1') + a_0(x)vy_1 = 0$$

Rearranging we want

$$v''y_1 + v'(2y_1' + a_1(x)y_1) + v(\underbrace{y_1'' + a_1(x)y_1' + a_0(x)y_1}_{0}) = 0$$

This reduces to needing to find v where

$$v''y_1 + v'(2y_1' + a_1(x)y_1) = 0$$

This becomes

$$\frac{v''}{v'} = \frac{-2y_1' - a_1(x)y_1}{y_1}$$

Which is

$$\frac{v''}{v'} = \frac{-2y_1'}{y_1} - a_1(x)$$

Integrating with respect to x gives

$$\ln(v') = -\ln(y_1^2) - \int a_1(x)\,dx$$

This gives

$$v' = e^{-\ln(y_1^2) - \int a_1(x)\,dx}$$

$$v' = \frac{1}{y_1^2} \cdot e^{-\int a_1(x)\,dx}$$

$$v = \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

Since $y_2 = y_1 v$ we get that

$$y_2 = y_1 \int \frac{e^{-a_1(x)\,dx}}{y_1^2}\,dx$$

Which is the formula we had given.

Note that y_1 and y_2 will be linearly independent because

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & v'y_1 + vy_1' \end{vmatrix}$$

$$= y_1 v' y_1 + y_1 v y_1' - y_1' v y_1$$

$$= y_1^2 v'$$

$$= e^{-\int a_1(x)\,dx} \neq 0$$

Summary: Let $a_1(x)$ be continuous on I.
Let y_1 be a solution to

$$y'' + a_1(x) y' + a_0(x) y = 0$$

on I where $y_1(x) \neq 0$ for all x in I.

Then,

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2}\, dx$$

will be another solution that is linearly independent with y_1